

Bäcklund transformations and Hamiltonian flows

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Abstract

In this work we show that, under given conditions, parametric Bäcklund transformations (BTs) for a finite dimensional integrable system can be interpreted as the solution to the equations of motion defined by an associated non-autonomous Hamiltonian. The two systems share the same constants of motion. This observation lead to the identification of the Hamiltonian interpolating the iteration of the discrete map defined by the transformations, that indeed will be a linear combination of the integrals appearing in the spectral curve of the Lax matrix. An application to the Toda periodic lattice is given.

KEYWORDS: Bäcklund transformations, integrable maps, spectrality property, Toda lattice

1 Introduction

In the last years there has been a huge interest in the theory of integrable discretization. With “integrable discretization” usually is meant (see e.g. [16]) a discrete analogue of an integrable system of equations (H being the Hamiltonian, possibly a function of all the N integrals of the model and $\{\cdot, \cdot\}$ being the Poisson brackets):

$$\dot{z}_i = f_i(z) = \{H, z_i\} \quad (1)$$

that is a map (implicit or explicit)

$$\tilde{z}_i = \Omega_i(\tilde{z}, z, \mu)$$

depending on some parameter μ such that approximates, at least at first order, the flow 1 (i.e. $\tilde{z}_i = z_i + \mu f_i(z) + O(\mu^2)$); that preserves, in the limit $\mu \rightarrow 0$, the Poisson structure $\{\cdot, \cdot\}$ and such that possesses N conserved quantities independent and in involution, possibly depending on μ but coinciding with the continuous ones in the limit $\mu \rightarrow 0$.

A great number of results, based on various methods, appeared in the literature in the last two decades on discretization of finite dimensional systems fitting the previous definition [1] [3] [2] [5] [10] [11] [12] [13] (see also [16] and references therein). Among the various techniques, the Bäcklund transformations approach appear to be noteworthy:

indeed, despite the fact to be “non-algorithmic” (unlike other methods, e.g. the “splitting method”), it possesses characteristics, such as explicitness, exact preservation of all the integrals and canonicity (see e.g. [7]) deserving special consideration.

The aim of this paper is to show how Bäcklund transformations for finite dimensional systems represent, provided the “spectrality property” [8] holds, an *exact* discretization of the underlying continuous integrable model. With the adjective *exact* we mean the equivalence of the trajectories, at all times and at all orders, of the discrete flow defined by the transformations and the trajectories of the continuous flow of the model.

In Section (2) we consider an integrable system described by a 2x2 Lax matrix possessing a set of (parametric) Bäcklund transformation with the spectrality property. We will show how these transformations can be seen as the integral curves of a system of non-autonomous Hamiltonian equations sharing the same conserved quantities with the ancestor model. Then it will be pointed out that indeed the recurrences defined by the transformations lie on the trajectories of the continuous system defined by the Hamiltonian

$$\mathcal{H} = \sum_k c_k H_k \quad (2)$$

where H_k are the conserved quantities of the continuous model. The constants c_k will depend on the free parameters appearing in the transformations. At the end of the Section we will discuss the “usefulness” of the formulae obtained.

In Section (3) we will give an application of what stated in Section (2) with a two-parameters BTs on the periodic Toda lattice. The last paragraph will present some numerical examples on this model.

2 BTs & integrable discretization: an Hamiltonian approach

In this Section we consider an integrable system with a 2x2 Lax matrix $L(\lambda)$, λ being the spectral parameter. The conserved quantities (independent and in involution) of the model can be obtained by the coefficients of some expansion in λ of the determinant and of the trace of $L(\lambda)$. So the characteristic equation:

$$\det(L(\lambda) - v\mathbb{1}) = 0 \quad \implies v^2 - \text{Tr}(L(\lambda))v + \det(L(\lambda)) = 0$$

defines the eigenvalues of $L(\lambda)$, v_1 and v_2 , in terms of the conserved quantities and λ . It is convenient, but not necessary, to think about our model as written in terms of canonical variables, say $\{p_i, q_i\}_{i=1}^N$, so that we have the usual Poisson brackets:

$$\{p_i, q_j\} = \delta_{ij} \quad \{p_i, p_j\} = \{q_i, q_j\} = 0$$

and the involutivity conditions on the conserved quantities H_i :

$$\{H_i, H_j\} = 0 \quad \forall i, j = 1 \dots N \quad (3)$$

A set of Bäcklund transformations for the system is simply a set of canonical transformations

$$\begin{cases} p_i \xrightarrow{BT} \tilde{p}_i = f_i(p, q) \\ q_i \xrightarrow{BT} \tilde{q}_i = g_i(p, q) \end{cases} \quad (4)$$

from $(p_i, q_i)_{i=1}^N$ to a new set $(\tilde{p}_i, \tilde{q}_i)_{i=1}^N$ such that N functions $(H_i)_{i=1}^N$ are invariant under the transformations, that is $H_i(\tilde{p}, \tilde{q}) = H_i(p, q)$ (here and after p and q stand for the sets $(p_i)_{i=1}^N$ and $(q_i)_{i=1}^N$ respectively). From this definition indeed the classical property of BTs (or better auto-BTs) to send solutions of the equations of motion into solutions is easily proved, since the two sets of canonically conjugate variables satisfy the same system of equations. The application of BTs in numerical analysis comes from the observation that actually the relations (4) define explicit recurrences just by thinking at tilded variables as the untilded ones but computed at the next time step (in the following notation p_i is equivalent to $(p_i)_0$, \tilde{p}_i is equivalent to $(p_i)_1$ and so on):

$$\begin{cases} (p_i)_{n+1} = f_i((p)_n, (q)_n) \\ (q_i)_{n+1} = g_i((p)_n, (q)_n) \end{cases} \quad (5)$$

As pointed out by Kuznetsov and Sklyanin [8], the invariance of the conserved quantities, and then of the spectrum of $L(\lambda)$, implies the existence of a *dressing* or *Darboux* matrix $D(\lambda)$ intertwining the two matrices, $L(\lambda)$ and $\tilde{L}(\lambda) \doteq L(\lambda, \tilde{p}, \tilde{q})$:

$$\tilde{L}(\lambda)D(\lambda) = D(\lambda)L(\lambda)$$

Also, because the transformations are canonical, it exists a generating function, say $F(q, \tilde{q})$, such that

$$\begin{cases} p_i = \frac{\partial F(q, \tilde{q})}{\partial q_i} \\ \tilde{p}_i = -\frac{\partial F(q, \tilde{q})}{\partial \tilde{q}_i} \end{cases}$$

Of particular interest are the BTs depending on a parameter, say μ . In this case one can always think that the transformations are connected to the identity, in the sense that for a value of the parameter, that can be always be chosen to be 0, one has $\tilde{p} = p$ and $\tilde{q} = q$. Indeed if the original transformations are not connected to the identity we can obtain a new set of transformations that, by construction, are the identity transformations for $\mu = 0$, by the following very simple argument (see also [7]).

Let the dressing matrix defining the transformations be $D(\lambda, \lambda_1)$ where λ_1 is the parameter of the transformations. Since BTs are canonical transformations we can compose a transformation with parameter λ_1 and a transformation with parameter λ_2 to obtain a two-parameters set of BTs. So composing the transformation with $\lambda_1 \doteq \lambda_0 + \mu$ with its inverse transformation but calculated at $\lambda_2 = \lambda_0 - \mu$ will give a transformation that in the limit $\mu \rightarrow 0$ goes to the identity. The only question is about the concrete invertibility of the map, but having the expression of $D(\lambda, \lambda_1)$ the inverse transformation is found just by the adjucate of the matrix $D(\lambda, \lambda_1)$, say $\hat{D}(\lambda, \lambda_1)$. The dressing matrix for the composed transformation then will be $D(\lambda, \lambda_0, \mu) = \hat{D}(\lambda, \lambda_2) \cdot D(\lambda, \lambda_1)$. We

will give an example of such construction in Section (3)

It is quite natural to interpret the parameter μ as a sort of “time” with respect to which the BTs evolve. As a matter of fact Bäcklund transformations are parametric canonical transformations; this means that they are the integral curves (whose parameter is μ) of an Hamiltonian vector field since, *at least locally*, there exist an Hamiltonian $K(\tilde{p}, \tilde{q}, \mu)$ such that (see for example [4] for the proof):

$$\begin{cases} \frac{\partial \tilde{q}_i}{\partial \mu} = \frac{\partial K}{\partial \tilde{p}_i} \\ \frac{\partial \tilde{p}_i}{\partial \mu} = -\frac{\partial K}{\partial \tilde{q}_i} \end{cases} \quad (6)$$

As now we will show, if the so-called *spectrality property*, introduced by Kuznetsov and Sklyanin [8] holds, then it is possible to find explicitly the function $K(\tilde{p}, \tilde{q}, \mu)$; this function is indeed related to the eigenvalues of the Lax matrix and so contains all the conserved quantities of the system, but it depends also on the parameter(s) of the transformations: BTs are then the integral curves of a non-autonomous Hamiltonian vector field (the time being μ).

In order to clarify the previous discussion let us recall what the spectrality property is. Suppose to have a set of BTs depending on a parameter μ :

$$\begin{cases} \tilde{p}_i = f_i(p, q, \mu) \\ \tilde{q}_i = g_i(p, q, \mu) \end{cases}$$

Assuming the inverse function theorem applies, we can say that there exists a generating function F such that:

$$\begin{cases} p_i = \frac{\partial F(q, \tilde{q}, \mu)}{\partial \tilde{q}_i} \\ \tilde{p}_i = -\frac{\partial F(q, \tilde{q}, \mu)}{\partial \tilde{q}_i} \end{cases}$$

The spectrality property says that (a function of) the canonically conjugate variable with respect to μ (say Φ) and μ itself lie on the spectral curve:

$$\Phi = \frac{\partial F}{\partial \mu} \Big|_{\tilde{q}=\tilde{q}(p,q,\mu)} \implies \det(L(\mu) - f(\Phi)\mathbb{1}) = 0 \quad (7)$$

It seems that this property is “universal” in the sense of being shared by a large class of models (Toda [15], Ruijsenaars [12], Henon-Heiles [6], Gaudin [7], Kirchhoff top [14], Mumford systems [9]...).

According to some authors ¹ it could be better understood by taking an algebraic, or geometric, point of view. Here we want just to give a mechanical (Hamiltonian) interpretation of this property. Our observation is very simple: the “variable” Φ is canonically conjugated to μ ; furthermore, when expressed in terms of the variables p

¹See e.g. [15] where the author writes “the property...needs more research to uncover its algebraic and geometric meaning”.

and q , it is a function of the conserved quantities because lies on the spectral curve. Enlarging the phase space by adding the couple (Φ, μ) we can write dF as the difference between two Poincaré Cartan forms:

$$dF = \sum_i p_i dq_i - \tilde{p}_i d\tilde{q}_i + \Phi d\mu \quad (8)$$

This means that if (p, q) are the integral curves generated by the Hamiltonian $T(p, q, \mu)$, then \tilde{p} and \tilde{q} are the integral curves generated by the Hamiltonian

$$K(\tilde{p}, \tilde{q}, \mu) = \hat{T}(p(\tilde{p}, \tilde{q}, \mu), q(\tilde{p}, \tilde{q}, \mu), \mu) + \Phi(\tilde{p}, \tilde{q}, \mu) \quad (9)$$

where \hat{T} is the transform of $T(p, q, \mu)$ when expressed in terms of \tilde{p} and \tilde{q} . Notice that the function Φ , thanks to spectrality, is invariant under the transformation, so $\hat{\Phi} = \Phi$. Now let us make this further observation: as explained some lines above, it is always possible to have a BTs connected to the identity, so that:

$$\tilde{p}_i|_{\mu=0} = p_i, \quad \tilde{q}_i|_{\mu=0} = q_i \quad (10)$$

Since we have the freedom to choose the function $T(p, q, \mu)$, we can choose it to be zero. With this choice $(p_i, q_i)_{i=1}^N$ are just constants (the initial conditions). From (9) and (10) then readily follows the following

Theorem 1 *Assuming the spectrality property (7) holds, the BTs $(\tilde{p}_i, \tilde{q}_i)_{i=1}^N$ are the integral curves of the following system of (non-autonomous) Hamiltonian equations:*

$$\begin{cases} \frac{\partial \tilde{q}_i}{\partial \mu} = \frac{\partial \Phi(\tilde{p}, \tilde{q}, \mu)}{\partial \tilde{p}_i} \\ \frac{\partial \tilde{p}_i}{\partial \mu} = -\frac{\partial \Phi(\tilde{p}, \tilde{q}, \mu)}{\partial \tilde{q}_i} \end{cases} \quad (11)$$

corresponding to the initial conditions (10).

By this point of view obtaining a set of explicit BTs transformations with the spectrality property for an integrable model means to have the explicit general solution of a non-autonomous Hamiltonian system sharing the same conserved quantities with the model. So we are considering two set of flows: the first set is given by all the flows defined by the integrable model, the second set is given by the flow defined by Φ . In the following we aim to clarify the relations among these flows in order to point out a concrete application of Theorem 1.

The key point is that the the flow defined by (11) possesses the same conserved quantities (3) as the corresponding integrable model defined by the Lax matrix $L(\lambda)$. Being the model integrable (by definition, in the sense of Liouville), there exist action-angle variables $(I_k, \theta_k)_{k=1}^N$, such that the action variables I_k are functions only of the conserved quantities and the couples (I_k, θ_k) are canonically conjugate [4]:

$$\{I_k, \theta_j\} = \delta_{kj}, \quad \{I_k, I_j\} = 0, \quad \{\theta_k, \theta_j\} = 0 \quad (12)$$

Actually we do not need that all the variables θ_j are really “angles”, it could be also a splitting among bounded and unbounded variables (say $\theta_k \in \mathbb{T}^m, k = 1 \dots m$ and $\theta_k \in \mathbb{R}^{N-m}, k = m+1 \dots N$). The key request is the canonicity (12) (obviously together with the conservation of each of I_j). Having this in mind, for the sake of simplicity we continue to use indistinctly the term “angle” in what follows.

Let us see what happens to the flow defined by Φ (11) in these coordinates. When expressed in terms of the set $(I_k, \theta_k)_{k=1}^N$ the Hamiltonian $\Phi(\tilde{p}, \tilde{q}, \mu) = \Phi(p, q, \mu)$ reduces to a function of action coordinates and μ only:

$$\Phi(p, q, \mu) = \check{\Phi}(I, \mu) = \check{\Phi}(\tilde{I}, \mu)$$

The system (11) then becomes:

$$\begin{cases} \frac{\partial \tilde{I}_j}{\partial \mu} = -\frac{\partial \check{\Phi}(\tilde{I}, \mu)}{\partial \tilde{\theta}_j} = 0 \\ \frac{\partial \tilde{\theta}_j}{\partial \mu} = \frac{\partial \check{\Phi}(\tilde{I}, \mu)}{\partial \tilde{I}_j} \Rightarrow \tilde{\theta}_j = \theta_j + \frac{\partial}{\partial \tilde{I}_j} \int_0^\mu \check{\Phi}(\tilde{I}, \mu) d\mu \end{cases} \quad (13)$$

In these coordinates it is clear what happens to the recurrences (5) defined by the BTs. Their explicit solution is indeed given by:

$$\begin{cases} (I_j)_n = (I_j)_0 \\ (\theta_j)_n = (\theta_j)_0 + n \frac{\partial}{\partial \tilde{I}_j} \int_0^\mu \check{\Phi}(\tilde{I}, \mu) d\mu \end{cases} \quad (14)$$

Replacing n by t we see that equations (14) represent the exact time discretization of the continuous system governed by the Hamiltonian:

$$\mathcal{H} = \int_0^\mu \check{\Phi} d\mu \quad (15)$$

Notice that there is no need to find explicitly the action-angle coordinates: the result (15) is independent of which coordinates we are expressing the equations of motion; we could also forget at its derivation and state simply the following

Theorem 2 *If the spectrality property holds, the discrete trajectories defined by the iteration of the recurrences*

$$\begin{cases} (p_i)_{n+1} = f_i(p_n, q_n, \mu) \\ (q_i)_{n+1} = g_i(p_n, q_n, \mu) \end{cases}$$

given by the BTs coincide with the trajectories of the Hamiltonian system defined by the Hamiltonian:

$$\mathcal{H} = \int_0^\mu \Phi d\mu$$

where Φ is defined by the spectrality property (7).

So, by one hand the BTs are the integral curves of the system of Hamiltonian equations generated by Φ , by the other hand they are the exact time discretization (i.e. preserving the trajectories) of the Hamiltonian equations generated by $\mathcal{H} = \int \Phi d\mu$. When expressed in canonical coordinates $(p_i, q_i)_{i=1}^N$, Φ is a function of the conserved quantities H_i (3) of the model: $\Phi = \Phi(H, \mu)$. The BTs are then the discretization of the flows corresponding to linear combinations of these conserved quantities; indeed for any function $\mathfrak{F}(p, q)$ on the phase space we can write:

$$\dot{\mathfrak{F}} = \{\mathcal{H}, \mathfrak{F}\} = \sum_k c_k \{H_k, \mathfrak{F}\}, \quad \text{where} \quad c_k = \frac{\partial}{\partial H_k} \int_0^\mu \Phi d\mu \quad (16)$$

Let us make some remarks about the “usefulness” of the previous formulae. Having in mind a physical model, usually one looks only at a discretization in some particular direction, corresponding to a given choice of the constants c_k appearing in (16). So the problem is about the freedom we have to choose the values of these constants.

As noted before, we always have the possibility to compose BTs writing products of dressing matrices with different parameters. The result will be a BTs depending on a set of free parameters. Also, the generating function F (8) and the spectral function Φ (and then the constants c_k themselves) will depend on these parameters. If there are enough parameters (that is N , the number of degrees of freedom of the system), the problem is then to invert the integrals for the constants c_k :

$$c_k = \frac{\partial}{\partial H_k} \int_0^\mu \Phi d\mu \quad (17)$$

For example for algebraic complete integrable systems one expects these integrals be hyperelliptic integrals. Notice however that for any fixed orbit and for a given set of the constants c_k , the integrals could be inverted numerically. An example will be given in the paragraph (3.2).

The parameter μ plays a role of a time not only for what concerns Theorem (1), but as it is clear also from (14), at smaller values of μ there corresponds smaller values in the time step discretization.

In the next Section we are going to present an application of the previous formulae.

3 An example from the Toda lattice

Let us consider the BTs of the periodic Toda lattice. The Lax matrix of the model is given by $L(\lambda) = l_N(\lambda) \cdots l_1(\lambda)$ where the local matrices $l_j(\lambda)$ are (see e.g. [15]):

$$l_j(\lambda) = \begin{pmatrix} \lambda + p_j & -e^{q_j} \\ e^{-q_j} & 0 \end{pmatrix} \quad (18)$$

The variables $(p_j, q_j)_{j=1}^N$ are couples of canonically conjugate variables. Also, for later use, it is useful to define the four functions $(A(\lambda), B(\lambda), C(\lambda), D(\lambda))$ as the elements of the lax matrix $L(\lambda)$:

$$L(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \quad (19)$$

The Poisson brackets among any combinations of these functions are defined by the following r -matrix structure:

$$\{L(\lambda), L(\mu)\} = [r(\lambda - \mu), L(\lambda) \otimes L(\mu)], \quad (20)$$

where $r(\lambda)$ is explicitly written as:

$$r(\lambda) = -\frac{1}{\lambda} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (21)$$

The corresponding Poisson brackets among the functions $(A(\lambda), B(\lambda), C(\lambda), D(\lambda))$ read as:

$$\{A(\lambda), A(\mu)\} = \{B(\lambda), B(\mu)\} = \{C(\lambda), C(\mu)\} = \{D(\lambda), D(\mu)\} = 0 \quad (22)$$

$$\{A(\lambda), B(\mu)\} = \frac{A(\lambda)B(\mu) - A(\mu)B(\lambda)}{\lambda - \mu}; \quad \{A(\lambda), C(\mu)\} = \frac{A(\mu)C(\lambda) - A(\lambda)C(\mu)}{\lambda - \mu} \quad (23)$$

$$\{A(\lambda), D(\mu)\} = \frac{C(\lambda)B(\mu) - C(\mu)B(\lambda)}{\lambda - \mu}; \quad \{B(\lambda), C(\mu)\} = \frac{D(\lambda)A(\mu) - D(\mu)A(\lambda)}{\lambda - \mu} \quad (24)$$

$$\{B(\lambda), D(\mu)\} = \frac{D(\lambda)B(\mu) - D(\mu)B(\lambda)}{\lambda - \mu}; \quad \{C(\lambda), D(\mu)\} = \frac{C(\lambda)D(\mu) - C(\mu)D(\lambda)}{\lambda - \mu} \quad (25)$$

Note that the determinant of each local Lax matrices $l_j(\lambda)$ is 1 so that the determinant of $L(\lambda) = A(\lambda)D(\lambda) - B(\lambda)C(\lambda)$ is also 1. The hyperelliptic curve defined by the characteristic polynomial $\det(v - L(\lambda)) = 0$, that is

$$v^2 - \text{tr}(L(\lambda))v + 1 = 0$$

is constant with respect to the Toda flows. All the commuting Hamiltonians H_j of the system can be obtained by the coefficients of the expansion of $\text{tr}(L(\lambda))$:

$$\text{tr}(L(\lambda)) = A(\lambda) + D(\lambda) = \sum_{k=0}^N H_k \lambda^k \quad (26)$$

Note that there are N Hamiltonians because of $H_N = 1$.

Following [15], we can construct a class of BTs for this system considering a similarity transformation through a matrix $D(\lambda)$ (the *Darboux matrix*) defining a map from the variables $\{(p_j, q_j)\}_{j=1}^N$ to the variables $\{(\tilde{p}_j, \tilde{q}_j)\}_{j=1}^N$:

$$L(\lambda, \tilde{p}, \tilde{q})D(\lambda) = D(\lambda)L(\lambda, p, q) \quad (27)$$

Due to the splitting $L(\lambda) = l_N(\lambda) \cdots l_1(\lambda)$ of the Lax matrix into local matrices, it is possible to recursively define a set of local matrices $D_j(\lambda)$ by the following relation:

$$l_j(\lambda, \tilde{p}, \tilde{q})d_j(\lambda) = d_{j+1}(\lambda)l_j(\lambda, p, q) \quad (28)$$

so that the transformation (27) is fulfilled by the position $D(\lambda) = d_1(\lambda)$. The simplest of the matrices d_j corresponds to the choice of linear dependence on the spectral parameter [15]:

$$d_j(\lambda) = \begin{pmatrix} \lambda - \zeta - \alpha_j \beta_j & -\alpha_j \\ \beta_j & 1 \end{pmatrix} \quad (29)$$

where ζ is believed to be a parameter and α_j and β_j depend on the dynamical variables. The dependences of α_j and β_j are immediately fixed by the off-diagonal part of (28), giving:

$$\begin{cases} \beta_{j+1} = e^{-\tilde{q}_j} \\ \alpha_j = e^{q_j} \end{cases} \quad (30)$$

With these positions, the similarity transformation (28) defines the following *implicit* map:

$$\begin{cases} \tilde{p}_j = - (e^{\tilde{q}_j - q_j} + e^{q_{j+1} - \tilde{q}_j} + \zeta) \\ p_j = - (e^{\tilde{q}_j - q_j} + e^{q_j - \tilde{q}_{j-1}} + \zeta) \end{cases} \quad (31)$$

Although implicit, it is very useful that the map has been written in this form in order to prove its canonicity. Indeed it can be easily checked that eqs. (31) can be found by the generating function $F(\zeta, q, \tilde{q})$ defined as follows:

$$p_j = \frac{\partial F}{\partial q_j}, \quad \tilde{p}_j = -\frac{\partial F}{\partial \tilde{q}_j}, \quad F \doteq \sum_k (\zeta(\tilde{q}_k - q_k) + e^{\tilde{q}_k - q_k} - e^{q_{k+1} - \tilde{q}_k}) \quad (32)$$

It is not an accident however that the matrices d_j (and hence $D = d_1$) have a determinant vanishing at $\lambda = \zeta$. In this case indeed the transformations (31) can be plainly inverted (in the sense to write *explicit* transformations). Indeed $D(\lambda)$ is given by:

$$D(\lambda) = \begin{pmatrix} \lambda - \zeta - \alpha_1 \beta_1 & -\alpha_1 \\ \beta_1 & 1 \end{pmatrix} \quad (33)$$

We already know that $\alpha_1 = e^{q_1}$ and $\beta_1 = e^{-\tilde{q}_N}$. But, since $\det(D(\zeta)) = 0$, $D(\zeta)$ has a (unique, up to an overall factor) kernel, given by $|\Omega\rangle = (1, -\beta_1)^T$. The combination of the unicity of this kernel with (27) gives that $|\Omega\rangle$ is also an eigenvector of $L(\zeta)$. Calling $v(\zeta)$ the corresponding eigenvalue, we have:

$$\begin{cases} A(\zeta) - \beta_1 B(\zeta) = v(\zeta) \\ C(\zeta) - \beta_1 D(\zeta) = -v(\zeta) \beta_1 \end{cases} \quad (34)$$

After the elimination of $v(\zeta)$ we are left with $\beta_1^2 B(\zeta) - \beta_1(A(\zeta) - D(\zeta)) - C(\zeta) = 0$ defining β_1 in terms of only the untilded dynamical variables. At this point the relation (27) gives *explicit* transformations.

So we have BTs with a free parameter and, as explained in Section (2), we can ask about the flow associated with this parameter. We remember that, since the map (31) defines canonical transformations, there exist (at least locally) an Hamiltonian $K(\tilde{p}, \tilde{q}, \zeta)$, such that

$$\begin{cases} \frac{\partial \tilde{q}}{\partial \zeta} = \frac{\partial K}{\partial \tilde{q}} \\ \frac{\partial \tilde{p}}{\partial \zeta} = -\frac{\partial K}{\partial \tilde{q}} \end{cases} \quad (35)$$

The flow defined by (31) is however not connected to the identity. In order to obtain a flow connected to the identity, as explained in Section (2), we can compose two BTs, the first with parameter $\zeta = \lambda_1$ and the second given by the inverse transformations of the first one but evaluated at $\zeta = \lambda_2$. It is clear then that when $\lambda_2 \rightarrow \lambda_1$ the identity map is recovered. So let us consider the inverse transformations of (31). They are defined by the inverse Darboux matrix D^{-1} but since (27) is homogeneous, we can take the adjugate matrix of D , say \hat{D} . So we have

$$L(\lambda, p, q) \hat{D}(\lambda) = \hat{D}(\lambda) L(\lambda, \tilde{p}, \tilde{q}) \quad (36)$$

with

$$\hat{D}(\lambda) = \begin{pmatrix} 1 & \alpha_1 \\ -\beta_1 & \lambda - \zeta - \alpha_1 \beta_1 \end{pmatrix} \quad (37)$$

We know that $\alpha_1 = e^{q_1}$ and $\beta_1 = e^{-\tilde{q}_N}$ but in order to have explicit transformations we need to write α_1 in terms of tilded variables. Note that $\hat{D}(\zeta)$ has a kernel, given by $|\hat{\Omega}\rangle = (\alpha_1, -1)^T$ that is also an eigenvector of $\tilde{L}(\zeta) \doteq L(\zeta, \tilde{p}, \tilde{q})$. So we have:

$$\begin{cases} \tilde{A}(\zeta)\alpha_1 - \tilde{B}(\zeta) = w(\zeta)\alpha_1 \\ \tilde{C}(\zeta)\alpha_1 - \tilde{D}(\zeta) = -w(\zeta) \end{cases} \quad (38)$$

After the elimination of $v(\zeta)$ we are left with $(\alpha_1)^2 \tilde{C}(\zeta) + \alpha_1(\tilde{A}(\zeta) - \tilde{D}(\zeta)) - \tilde{B}(\zeta) = 0$. Note also that since $\det \tilde{L} = \det L = 1$ there are two possibilities for the eigenvalue $w(\zeta)$: or $w(\zeta) = v(\zeta)$ or $w(\zeta) = v^{-1}(\zeta)$, where $v(\zeta)$ is defined by (34). In order to have the inverse transformations of the previous one the correct choice is $w(\zeta) = v^{-1}(\zeta)$ as now we will show. This will fix the relative sign of the square roots appearing in the solutions of the quadratic equations defining β_1 and α_1 .

The element (1, 1) of (36) gives $A(\zeta) - \beta_1 B(\zeta) = \tilde{A}(\zeta) + \alpha_1 \tilde{C}(\zeta)$. But from (34) we have $A(\zeta) - \beta_1 B(\zeta) = v(\zeta)$. Now is simple to see that the equivalence $w(\zeta) = v^{-1}(\zeta)$, or more explicitly:

$$(\tilde{D}(\zeta) - \alpha_1 \tilde{C}(\zeta))(\tilde{A}(\zeta) + \alpha_1 \tilde{C}(\zeta)) = 1$$

is just the equation defining α_1 because of $\tilde{A}\tilde{D} = \tilde{B}\tilde{C} + 1$.

So we are ready to compose the two maps. We can define a new Darboux matrix $\mathcal{D}(\lambda, \lambda_1, \lambda_2)$ given by the product $\hat{D}(\lambda, \lambda_2)D(\lambda, \lambda_1)$:

$$\begin{aligned} \tilde{L}D(\lambda, \lambda_1) &= D(\lambda, \lambda_1)L, & \tilde{\tilde{L}}\hat{D}(\lambda, \lambda_2) &= \hat{D}(\lambda, \lambda_2)\tilde{L}, \longrightarrow \\ &\longrightarrow \tilde{\tilde{L}}\hat{D}(\lambda, \lambda_2)D(\lambda, \lambda_1) &= \hat{D}(\lambda, \lambda_2)D(\lambda, \lambda_1)L \end{aligned} \quad (39)$$

with:

$$\hat{D}(\lambda, \lambda_2) = \begin{pmatrix} 1 & x_2 \\ -y_2 & \lambda - \lambda_2 - x_2 y_2 \end{pmatrix} \quad D(\lambda, \lambda_1) = \begin{pmatrix} \lambda - \lambda_1 - x_1 y_1 & -x_1 \\ y_1 & 1 \end{pmatrix} \quad (40)$$

From the relations (39) one has $y_1 = y_2 = e^{-\tilde{q}_N}$. So we set $y_1 = y_2 \doteq y$. Also the product $\hat{D}(\lambda, \lambda_2)D(\lambda, \lambda_1)$ depends only on the difference $x_1 - x_2$, so we set $x_1 - x_2 \doteq x$. The result is:

$$\mathcal{D}(\lambda, \lambda_1, \lambda_2) = \begin{pmatrix} \lambda - \lambda_1 - xy & -x \\ y(\lambda_1 - \lambda_2 + xy) & \lambda - \lambda_2 + xy \end{pmatrix} \quad (41)$$

From now on we consider directly the transformations given by this Darboux matrix, so omitting the intermediate step we look at:

$$\tilde{L}(\lambda)\mathcal{D}(\lambda, \lambda_1, \lambda_2) = \mathcal{D}(\lambda, \lambda_1, \lambda_2)L(\lambda) \quad (42)$$

The matrix (41) is obviously degenerate for $\lambda = \lambda_1$ and $\lambda = \lambda_2$. The respective kernels are given by $|\Omega_1\rangle = (1, -y)^T$ and $|\Omega_2\rangle = (x, \lambda_2 - \lambda_1 - xy)^T$. These are also eigenvectors of $L(\lambda_1)$ and $L(\lambda_2)$ with eigenvalues $v(\lambda_1)$ and $v^{-1}(\lambda_2)$:

$$L(\lambda_1)|\Omega_1\rangle = v(\lambda_1)|\Omega_1\rangle, \quad L(\lambda_2)|\Omega_2\rangle = v^{-1}(\lambda_2)|\Omega_2\rangle \quad (43)$$

We can explicitly write:

$$\begin{aligned} v(\lambda_1) &= \frac{A(\lambda_1) + D(\lambda_1) - \gamma(\lambda_1)}{2}, & v(\lambda_2) &= \frac{A(\lambda_2) + D(\lambda_2) + \gamma(\lambda_2)}{2} \\ \text{with } \gamma(\lambda)^2 &= (A(\lambda) + D(\lambda))^2 - 4 \end{aligned} \quad (44)$$

From relations (43) the variables x and y are then easily found:

$$\begin{cases} y = \frac{A(\lambda_1) - D(\lambda_1) + \gamma(\lambda_1)}{2B(\lambda_1)} \\ x = \frac{2B(\lambda_1)B(\lambda_2)(\lambda_2 - \lambda_1)}{B(\lambda_2)(A(\lambda_1) - D(\lambda_1) + \gamma(\lambda_1)) - B(\lambda_1)(A(\lambda_2) - D(\lambda_2) - \gamma(\lambda_2))} \end{cases} \quad (45)$$

The relations above enable us to obtain in explicit form the BTs. Posing

$$\lambda_1 = \lambda_0 + \mu, \quad \lambda_2 = \lambda_0 - \mu \quad (46)$$

we see that in the limit $\mu \rightarrow 0$ the identity transformation is recovered.

3.1 Generating functions and Hamiltonian flows

In order to find the generating function for the composed transformation produced by the Darboux matrix (41) it is possible to proceed as for the case of one parameter transformations. So, let us find the *implicit* transformations $\tilde{p} = \tilde{p}(q, \tilde{q}, \mu)$ and $p = p(q, \tilde{q}, \mu)$ in order to obtain the generating function F as the solution of the system:

$$p_j = \frac{\partial F}{\partial q_j}, \quad \tilde{p}_j = -\frac{\partial F}{\partial \tilde{q}_j} \quad (47)$$

Again we look at the relation among the local Lax matrices:

$$l_j(\lambda, \tilde{p}, \tilde{q})\mathfrak{d}_j(\lambda) = \mathfrak{d}_{j+1}(\lambda)l_j(\lambda, p, q) \quad (48)$$

where now the matrices \mathfrak{d}_j are given by:

$$\mathfrak{d}_j(\lambda) = \begin{pmatrix} \lambda - \lambda_1 - x_j y_j & -x_j \\ y_j(\lambda_1 - \lambda_2 + x_j y_j) & \lambda - \lambda_2 + x_j y_j \end{pmatrix} \quad (49)$$

Recall that $\mathfrak{d}_1 = \mathcal{D}$ so that $x_1 = x$ and $y_1 = y$. From the off-diagonal elements of (48) we obtain the expressions of x_j and y_j in terms of the dynamical variables q_k and \tilde{q}_k as:

$$x_j = e^{q_j} - e^{\tilde{q}_j}, \quad y_{j+1}(2\mu + x_{j+1}y_{j+1}) = e^{-\tilde{q}_j} - e^{-q_j} \quad (50)$$

It is useful to introduce the variable η_j defined by:

$$\eta_j \doteq x_j y_j + \mu$$

Indeed from:

$$x_{j+1}y_{j+1}(2\mu + x_{j+1}y_{j+1}) = (e^{q_{j+1}} - e^{\tilde{q}_{j+1}})(e^{-\tilde{q}_j} - e^{-q_j}) \doteq w_{j+1}$$

We see that $\eta_j^2 = \mu^2 + w_j$. With the help of these relations, the remaining equations (48) give:

$$\begin{cases} \tilde{p}_j = -\lambda_0 - \frac{e^{\tilde{q}_j}\eta_j + e^{q_j}\eta_{j+1}}{e^{q_j} - e^{\tilde{q}_j}} \\ p_j = -\lambda_0 - \frac{e^{\tilde{q}_j}\eta_{j+1} + e^{q_j}\eta_j}{e^{q_j} - e^{\tilde{q}_j}} \end{cases}$$

After some calculations it is not difficult to check that the generating function F for these transformations is given by:

$$F = \sum_k \left(\mu \ln \left(\frac{\eta_k + \mu}{\eta_k - \mu} \right) - 2\eta_k - \lambda_0(q_k - \tilde{q}_k) \right) \quad (51)$$

Now, as explained in Section 2, we can consider an extension of the phase space defining the variable conjugate to μ :

$$\Phi = \frac{\partial F}{\partial \mu} = \sum_k \ln \left(\frac{\eta_k + \mu}{\eta_k - \mu} \right) \quad (52)$$

By an Hamiltonian point of view Φ , when evaluated at $\tilde{q} = \tilde{q}(p, q, \mu)$, is no other than the new Hamiltonian flow defined by the canonical transformations engendered by the generating function F . So if the variables $(p_i, q_i)_{i=1}^N$ represent the integral curves generated by an Hamiltonian $T(p, q, \mu)$, the variables $(\tilde{p}_i, \tilde{q}_i)_{i=1}^N$ are the integral curves generated by the new Hamiltonian:

$$K(\tilde{p}, \tilde{q}, \mu) = \hat{T}(\tilde{p}, \tilde{q}, \mu) + \hat{\Phi}(\tilde{p}, \tilde{q}, \mu) \quad (53)$$

Also, taking $\hat{T} = 0$, we can consider the values of $(p_i, q_i)_{i=1}^N$ as the set of our initial values; the BTs are then the general solution of the system of equations given by the Hamiltonian $\hat{\Phi}$.

The function $\hat{\Phi}(\tilde{p}, \tilde{q}, \mu)$ assumes the same value when evaluated at (\tilde{p}, \tilde{q}) or (p, q) due to the equivalence $\tilde{A}(\lambda) + \tilde{D}(\lambda) = A(\lambda) + D(\lambda)$ (recall that $A + D$ is the generating function of *all* the conserved quantities of the system, see eq. (26)). So

$$\hat{\Phi}(p, q, \mu) = \Phi(q, \tilde{q}(p, q, \mu), \mu) \quad (54)$$

At this point it is convenient to calculate explicitly the function $\hat{\Phi}$ in (53). From (52) we have that:

$$\Phi(q, \tilde{q}, \mu) = \sum_k \ln \left(\frac{\eta_k + \mu}{\eta_k - \mu} \right)$$

In order to write Φ explicitly in terms of $(p_i, q_i)_{i=1}^N$ we note that the matrices \mathfrak{d}_j (49) are degenerate when $\lambda = \lambda_1$ and $\lambda = \lambda_2$. The kernels of $\mathfrak{d}_j(\lambda_1)$ are given by $|\omega_j(\lambda_1) \rangle = (x_j, \mu - \eta_j)^T$ and the relations (48) give:

$$l_j(\lambda_1) |\omega_j(\lambda_1) \rangle = \nu_{j+1}^1 |\omega_{j+1}(\lambda_1) \rangle$$

for some function ν_{j+1}^1 . More explicitly:

$$(p_j + \lambda_1)x_j - e^{q_j}(\mu - \eta_j) = \nu_{j+1}^1 x_{j+1} \quad (55)$$

But from the combination of the elements (2, 1) and (2, 2) of (48) we have:

$$(p_j + \lambda)x_j = e^{q_j}(\lambda - \lambda_0 - \eta_j) + e^{\tilde{q}_j}(\lambda - \lambda_0 + \eta_{j+1}) \quad (56)$$

Evaluating the above equation at $\lambda = \lambda_1$ and confronting with (55) leads to the identification of the function ν_{j+1}^1 as:

$$\nu_{j+1}^1 = \frac{e^{\tilde{q}_j}(\mu + \eta_{j+1})}{x_{j+1}}$$

Repeating the same argument for the kernels $|\omega_j(\lambda_2) \rangle = (-x_j, \mu + \eta_j)^T$ of $\mathfrak{d}_j(\lambda_2)$, we can write for some function ν_{j+1}^2 :

$$l_j(\lambda_2) |\omega_j(\lambda_2) \rangle = \nu_{j+1}^2 |\omega_{j+1}(\lambda_2) \rangle$$

and again with the help of (56) evaluated at $\lambda = \lambda_2$ we find:

$$\nu_{j+1}^2 = \frac{e^{\tilde{q}_j}(\eta_{j+1} - \mu)}{x_{j+1}}$$

Now due to the factorization of $L(\lambda)$ in local matrices it is straightforward to show that the products $\prod_k \nu_k^1$ and $\prod_k \nu_k^2$ are eigenvalues of $L(\lambda_1)$ and $L(\lambda_2)$ respectively. But from the analysis of the eigenvalue of $L(\lambda_1)$ and $L(\lambda_2)$ of the previous section we know that, when expressed in terms of the untilded variables we have (see (43) and (44)):

$$L(\lambda_1) |\Omega_1 \rangle = v(\lambda_1) |\Omega_1 \rangle, \quad L(\lambda_2) |\Omega_2 \rangle = v^{-1}(\lambda_2) |\Omega_2 \rangle \quad (57)$$

Collecting all together we see that $\frac{\partial F}{\partial \mu}$ as in (52) is given by:

$$\begin{aligned} \Phi(p, q, \mu) &= \left. \frac{\partial F}{\partial \mu} \right|_{\tilde{q}=\tilde{q}(p, q, \mu)} = \ln \left(\frac{\prod_k \nu_k^1}{\prod_k \nu_k^2} \right) \Big|_{\tilde{q}=\tilde{q}(p, q, \mu)} = \ln(v(\lambda_1)v(\lambda_2)) = \\ &= \ln \left(\frac{A(\lambda_1) + D(\lambda_1) - \gamma(\lambda_1)}{2} \frac{A(\lambda_2) + D(\lambda_2) - \gamma(\lambda_2)}{2} \right) \end{aligned} \quad (58)$$

This is the spectrality property of Kuznetsov and Sklyanin (see e.g. [8]): a function of the canonically conjugate variable with respect to the parameter of the transformations and the parameter itself lie on the spectral curve: $\det(f(v) - L(\mu)) = 0$. The result (58) can be also rewritten in terms of inverse hyperbolic function $\operatorname{arccosh}(\cdot)$:

$$\left. \frac{\partial F}{\partial \mu} \right|_{\tilde{q}=\tilde{q}(p,q,\mu)} = -\operatorname{arccosh}\left(\frac{A(\lambda_1) + D(\lambda_1)}{2}\right) - \operatorname{arccosh}\left(\frac{A(\lambda_2) + D(\lambda_2)}{2}\right) \quad (59)$$

The BTs given by (41) are then the integral curves of the flow generated by the Hamiltonian (59).

As an example let us take the simplest case, that is $N = 2$.

The trace of the Lax matrix is now given by:

$$A(\lambda) + B(\lambda) = \lambda^2 + (p_1 + p_2)\lambda + p_1 p_2 - 2\cosh(q_1 - q_2) \quad (60)$$

In this case it is also quite straightforward to find explicitly a set of action-angle variables. A choice is given by the following relations:

$$\begin{cases} I_1 = \frac{p_1 + p_2}{2}, & I_2 = 2\cosh(q_1 - q_2) + \left(\frac{p_1 - p_2}{2}\right)^2 \\ \theta_1 = q_1 + q_2, & \theta_2 = \frac{F\left(\frac{p_2 - p_1}{2\sqrt{I_2 - 2}}, k\right)}{\sqrt{I_2 + 2}} \end{cases} \quad (61)$$

where $F(z, k)$ is the complete elliptic integral of the first kind and the modulus k is given by $k^2 = \frac{I_2 - 2}{I_2 + 2}$. Now expression (60) is written as $A(\lambda) + B(\lambda) = (\lambda + I_1)^2 - I_2$. According to Theorem (2) the recurrences defined by the BTs (5) discretize the continuous flow corresponding to the Hamiltonian:

$$\mathcal{H} \stackrel{N=2}{=} \int_{\lambda_0 + \mu}^{\lambda_0 - \mu} \operatorname{arccosh}\left(\frac{(\lambda + I_1)^2 - I_2}{2}\right) d\lambda$$

so that

$$\begin{cases} \tilde{\theta}_1 = \theta_1 + \frac{\partial}{\partial I_1} \int_{\lambda_0 + \mu}^{\lambda_0 - \mu} \operatorname{arccosh}\left(\frac{(\lambda + I_1)^2 - I_2}{2}\right) d\lambda \\ \tilde{\theta}_2 = \theta_2 + \frac{\partial}{\partial I_2} \int_{\lambda_0 + \mu}^{\lambda_0 - \mu} \operatorname{arccosh}\left(\frac{(\lambda + I_1)^2 - I_2}{2}\right) d\lambda \end{cases} \quad (62)$$

giving

$$\begin{cases} \tilde{\theta}_1 = \theta_1 + \operatorname{arccosh}\left(\frac{(\lambda_0 - \mu + I_1)^2 - I_2}{2}\right) - \operatorname{arccosh}\left(\frac{(\lambda_0 + \mu + I_1)^2 - I_2}{2}\right) \\ \tilde{\theta}_2 = \theta_2 + \frac{1}{\sqrt{I_2 + 2}} \left(F\left(\frac{\lambda_0 - \mu + I_1}{\sqrt{I_2 - 2}}, k\right) - F\left(\frac{\lambda_0 + \mu + I_1}{\sqrt{I_2 - 2}}, k\right) \right) \end{cases} \quad (63)$$

In these coordinates the BTs are linearized. Indeed if we denote, as in (14), with $(I_j)_n$ and $(\theta_j)_n$ the n^{th} iterates of the variables I_j and θ_j , we obtain from (63):

$$\begin{cases} (I_1)_n = (I_1)_0 \\ (I_2)_n = (I_2)_0 \\ (\theta_1)_n = (\theta_1)_0 + n \operatorname{arccosh} \left(\frac{(\lambda_0 - \mu + I_1)^2 - I_2}{2} \right) - n \operatorname{arccosh} \left(\frac{(\lambda_0 + \mu + I_1)^2 - I_2}{2} \right) \\ (\theta_2)_n = (\theta_2)_0 + \frac{n}{\sqrt{I_2 + 2}} \left(F\left(\frac{\lambda_0 - \mu + I_1}{\sqrt{I_2 - 2}}, k\right) - F\left(\frac{\lambda_0 + \mu + I_1}{\sqrt{I_2 - 2}}, k\right) \right) \end{cases} \quad (64)$$

Also, confronting with the definition of θ_2 (61), it becomes clear that the BTs correspond to addition formulae for elliptic (or hyper-elliptic in the case $N > 2$) integrals. Indeed we can write:

$$F(\tilde{P}, k) = F(P, k) + F(\lambda_-, k) - F(\lambda_+, k) \quad (65)$$

where $\tilde{P} = \frac{\tilde{p}_2 - \tilde{p}_1}{2\sqrt{I_2 - 2}}$, $P = \frac{p_2 - p_1}{2\sqrt{I_2 - 2}}$ and $\lambda_{\pm} = \frac{\lambda_0 \pm \mu + I_1}{\sqrt{I_2 - 2}}$. For analogous results on addition formulae for Weierstraß \wp function or Jacobi elliptic functions see respectively [9] and [14].

3.2 Numerics

The results of Section (3) together with Theorem (2) lead to the following result: the BTs for the Toda discretize the eqs. of motion given by the following Hamiltonian:

$$\mathcal{H} = - \int_{\lambda_0 - \mu}^{\lambda_0 + \mu} \operatorname{arccosh} \left(\frac{\operatorname{Tr}(L(\lambda))}{2} \right) d\lambda \quad (66)$$

where we recall that the trace of $L(\lambda)$ is the generating function of all the integrals:

$$\operatorname{tr}(L(\lambda)) = A(\lambda) + D(\lambda) = \sum_{k=0}^N H_k \lambda^k, \quad H_N = 1 \quad (67)$$

so that the corresponding continuous eqs. of motion, for any function \mathfrak{F} of the phase space, is:

$$\dot{\mathfrak{F}} = \{\mathcal{H}, \mathfrak{F}\} = \sum_k c_k \{H_k, \mathfrak{F}\}, \quad \text{where } c_k \doteq - \int_{\lambda_0 - \mu}^{\lambda_0 + \mu} \frac{\lambda^k}{\sqrt{\operatorname{Tr}(L(\lambda))^2 - 4}} d\lambda \quad (68)$$

Let us take the explicit case $N = 3$. The three integrals of motion are given by:

$$\begin{cases} H_1 = p_1 + p_2 + p_3 \\ H_2 = p_1 p_2 + p_2 p_3 + p_3 p_1 - e^{q_1 - q_3} - e^{q_2 - q_1} - e^{q_3 - q_2} \\ H_3 = p_1 p_2 p_3 - p_2 e^{q_1 - q_3} - p_3 e^{q_2 - q_1} - p_1 e^{q_3 - q_2} \end{cases} \quad (69)$$

Given the formulae (68) it is possible to plot the continuous flow corresponding to some

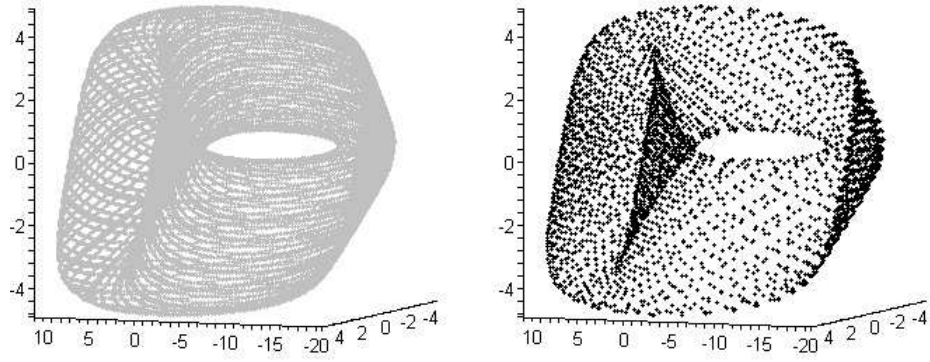


Figure 1: The initial conditions are $p_1 = -20$, $p_2 = 10$, $p_3 = 1$, $q_1 = 1$, $q_2 = 0$, $q_3 = 0.9$. The corresponding values for the constants c_k are $c_0 = -0.005$, $c_{12} = -0.048$ and $c_2 = -0.512$.

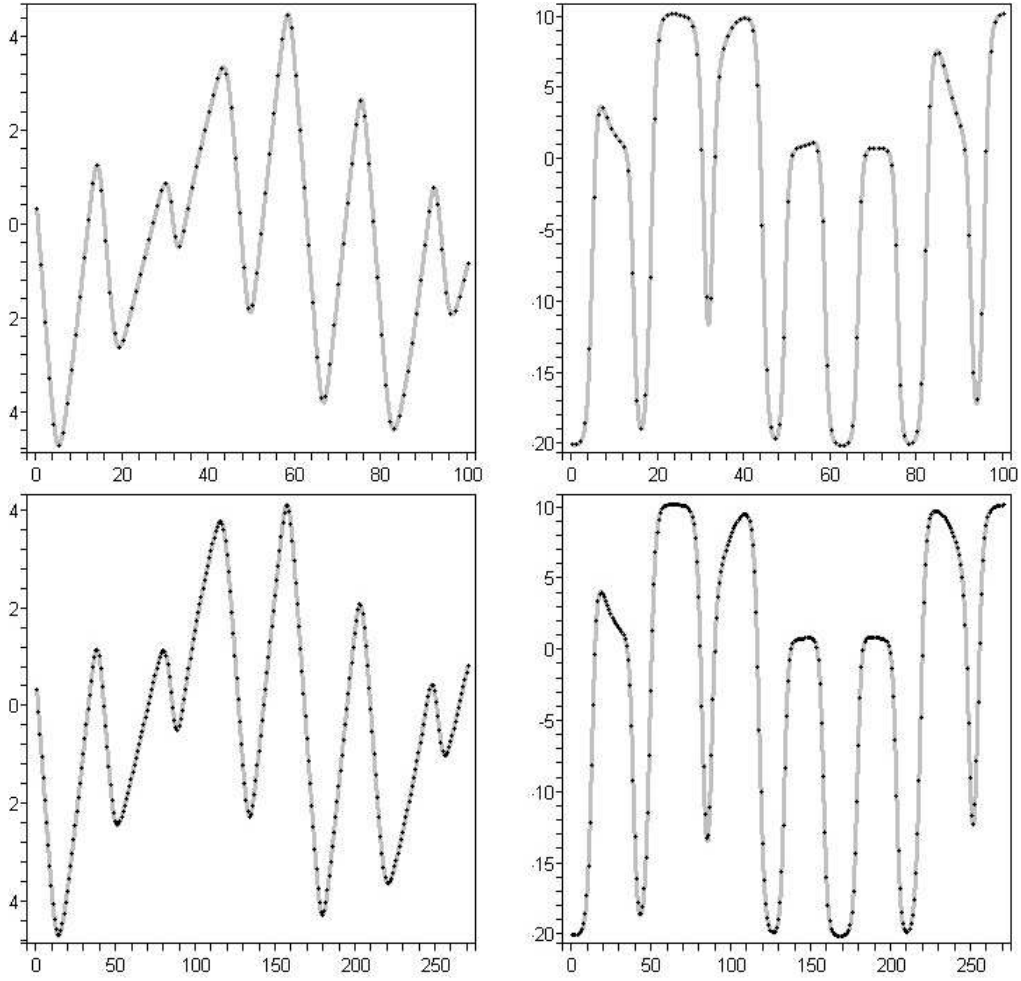


Figure 2: The values of μ are 5 (top) and 2 (bottom). The corresponding values for the constants (c_0, c_1, c_2) are $(-0.005, -0.048, -0.512)$ and $(-0.0018, -0.0183, -0.1849)$

choice of the constants c_k and the corresponding discrete flow given by the recurrences defined by the explicit BTs (from (42), (45) and (46)). In order to make a comparison one has to choose the same initial conditions for the two flows. Once they are fixed (so that the conserved quantities H_1 , H_2 and H_3 are given), it is possible to calculate, numerically or explicitly, the values of the constants c_k knowing those of λ_0 and μ . Conversely, if there were three free parameters instead of two in the BTs, it would be possible to calculate their values knowing those of the constants c_k . Let in our case choose the values $\lambda_0 = 10$ and $\mu = 5$. A plot of the continuous flow given by (15) is given in figure (1) on the left. On the right there are the first 3500 iterations of the BTs with the same initial conditions (both initial conditions and the values of the c_k 's are reported in the caption). The variables on the axes are $(p_3, q_2 - Q, q_3 - Q)$, where the center of mass $Q = \frac{q_1 + q_2 + q_3}{3}$ corresponding to a linear motion, has been subtracted to reveal the structure of a 2-torus, in accordance with the Liouville-Arnold theorem. In figure (2) we report an example of the continuous trajectories (in grey) and the discrete ones (black dots): the plots on the top correspond to the variables $q_1 - Q$ (on the right) and p_1 (on the left): the values of the initial conditions and of the parameters λ_0 and μ used are the same as for figure (1). For the two plots on the bottom we changed only the value of μ from 5 to 2: as explained in Section (2) and as can be seen from the figures, to a smaller value of the parameter μ it corresponds a smaller value of the time step size.

Acknowledgments

I wish to thank Andrew Hone for useful discussions, suggestions and remarks. Also I wish to acknowledge the financial support of the "Istituto Nazionale di Alta Matematica" (Italian National Institute for High Mathematics) as an INdAM-COFUND Marie Curie scholarship holder.

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